PERFECT NUMBERS AND PYTHAGOREAN TRIPLES

JONATHAN S. BLOOM

Abstract:

During the summer of 1999 I spent ten weeks solving problems published in the American Mathematical Monthly journal. Of the problems I attempted to solve, I completed three successfully and have submitted them to a mathematics journal to be considered for publication. The first problem I solved dealt with perfect numbers, which are numbers the sum of whose proper divisors equals the number, such as the number 6. This problem asked to prove that two consecutive numbers cannot both be perfect. After several approaches, I was able to solve the problem by looking at the sum of the divisors (mod 3). The next problem solved had to do with Pythagorean triples. This problem asked to prove that there exist infinitely many pairs of triples that differ by 3 or 4 in each coordinate. The solution I found, although not as elegant as the perfect number solution, did rest on a branch of mathematics that was new to me. The key in this problem was realizing, after manipulating the equations for generating triples, that there exist an infinite number of solutions of the Pell equation.

Theorem:

Show that there are infinitely many pairs ((a, b, c) (a', b', c')) of primitive Pythagorean triples such that $|a - a'|$, $|b - b'|$, $|c - c'|$ are all equal to 3 or 4.

Proof:

Let (x, y, z), (x', y', z') be Pythagorean triples such that $x' = x+3$, $y'=y+3$, $z'=z+4$. Since Pythagorean triples have the property that $x'^2+y'^2=z'^2$, we can write $(x+3)^2+(y+3)^2=(z+4)^2$ which yields $3x+3y+9 = 4z+8$*.

By the theorem on generation of primitive Pythagorean triples we know that $x=2mn$, $y=m^2-n^2$, $z=m^2+n^2$. By substituting these values into (*) we get: $m^2-6mn+7n^2-1=0$. Using the quadratic formula we get that $m=3n\pm\sqrt{(2n^2+1)}$. Since we are looking for whole numbers we need $\sqrt{(2n^2+1)} = s$, $s\in N$. So we have $s^2= 2n^2+1$ or $1= s^2- 2n^2$ which is a Pell equation and it is known that there are infinitely many distinct integer solutions to this equation.

We need to verify that m and n are opposite parity. First we see that n must contain a 2 since $1= s^2- 2n^2$, therefore $s^2- 2n^2 \equiv 1 \pmod 4$. Since $s^2= 2n^2+1$ we know that s is odd and therefore $s^2 \equiv 1 \pmod 4$. From this it is clear that n must be even.

Next we must show that $(m, n)=1$. First, we see that if $d \mid s$ and $d \mid n$ then since $s^2= 2n^2+1$, $d \mid 1$. Therefore $(s, n)=1$. Now let $d \mid m$ and $d \mid n$. We know that $m=3n+\sqrt{(2n^2+1)}$ and if $d \mid n$ then $d \mid \sqrt{(2n^2+1)}$. So it is clear that $d \mid s$ and since $(s, n)=1$ then $(m, n)=1$.
$=(2s+n)$ and $q=(s+n)$ we can conclude that $(p, q)=1$.

Next we generate (x', y', z') from the (m, n) pair. We know that $x'=2mn+3$, $y'=m^2-n^2+3$, $z'=m^2+n^2+4$. By letting $m=3n+\sqrt{(2n^2+1)}$ and multiplying out we get that $x'=6n^2+2n\sqrt{(2n^2+1)}+3$, $y'=10n^2+6n\sqrt{(2n^2+1)}+4$, $z'=12n^2+6n\sqrt{(2n^2+1)}+5$. Next we substitute s^2 for (n^2+1) and s for $(2n^2+1)$ to get $x'=3s^2+2ns=(2s+n)^2-(s+n)^2$, $y'=2(2s+n)(s+n)$, $z'=5s^2+6ns+2n^2=(2s+n)^2+(s+n)^2$. From this we can see that $p=(2s+n)$ and $q=(s+n)$.

To show that (x', y', z') is a primitive tirple we first need to show that p and q are of opposite parity. We know that $p=(2s+n)$ and $q=(s+n)$. Since we know n must be even and s is odd by definition it is clear that p is always even and q is always odd.

Next we need to show that $(p, q)=1$. First assume that $d \mid p$ and $d \mid q$. Therefore d must divide the difference so $d \mid p-q$ or $d \mid s$. If $d \mid s$ and by assumption we know that $d \mid (s+n)$ we see that $d \mid n$. But we know $(n, s)=1$ so $d=1$ and $(p, q)=1$.

Theorem:

A natural number is perfect if it is the sum of its proper divisors. Prove that two consecutive numbers cannot both be perfect.

Proof:

Assume both n and n+1 are perfect. From Euler we know that all even perfect numbers are of the form $2^{p-1}(2^p-1)$. Since neither 5 nor 7 is perfect we can let $p>2$. Therefore the even member of the pair $(n, n+1)$ is $\equiv 0 \pmod 4$. From Euler's theory on odd perfect numbers we know that any odd perfect number is $\equiv 1 \pmod 4$. From this we can conclude that n must be even and n+1 odd.

Since n is of the form $2^{p-1}(2^p-1)$ and $p>2$ we see that $n\equiv 1 \pmod 3$ and $n+1\equiv 2 \pmod 3$. Since we assume n+1 to be perfect then $\sigma(n+1)=2n+2$ so $\sigma(n+1) \equiv 1 \pmod 3$.

Now let $d_1 \mid n+1$ and $d_2 \mid n+1$ such that $d_1*d_2 = n+1$. So $d_1*d_2 \equiv 2 \pmod 3$.

Without loss of generality let $d_1\equiv 1 \pmod 3$ and $d_2\equiv 2 \pmod 3$. Now by adding all the divisors of n+1 (mod 3) we get the sum $1+2+1+2+\ldots+1+2$, so $\sigma(n+1) \equiv 0 \pmod 3$, a contradiction.