# A SOLAR MODEL

**Michael Fine**

**Abstract:** The problem of stellar structure is quite complicated and would be a computational nightmare to solve without the help of some computational software. In this study, Mathcad 2000 Professional was utilized to construct a numerical model. The equations of stellar structure used are both static (time independent) and spherically symmetric, and the Sun's strong magnetic field was neglected. Both radiative transfer and convection were considered, and a model for stellar opacity was developed. Utilizing the results, the Sun's moment of inertia was found. A 60 solar mass star was also modeled, and the results were compared with that of the Sun.

## I. INTRODUCTION

Why do we study the sun? Without the sun, life would cease to exist. For centuries, the Sun has been worshipped, feared, studied, and analyzed. Both the Greeks and Romans worshipped Apollo, who was fabled to ride across the sky on a fiery chariot drawn by wild horses. The Egyptians worshipped the Sun through Ra. The Japanese call their country Nippon, which means "Land of the Rising Sun". Various cultures have interpreted eclipses as a sign of impending disaster. Ancient legends speak of celestial monsters devouring the sun. "For Norse tribes, it was a wolf called Skoll. In Vietnam, it was a giant frog, in Argentina a jaguar, and in Siberia a vampire" (Morse). The Sun has also influenced artists of all mediums. Painters have mimicked the Sun's vivid colors during sunrise or sunset on canvas, while composers have captured their mood through music.

The Sun has played an important role in shaping world cultures. Today we seek to better understand how the Sun works, why it changes, and how these changes influence the Earth. Modern technology allows us to tackle these problems better than ever before. In this paper, I will present a model of the Sun's interior that will help answer some of these tough questions.

## II. THE MODEL

The basic equations of stellar structure, those governing how pressure, mass, luminosity, and temperature change with radius, are as follows:

Stars are held together by gravity, which tries to compress everything to the center. What holds an ordinary star up, preventing total collapse, is thermal and radiation pressure. Both pressures try to expand the star outward to infinity. In any given layer of a star, there is a `balance` between the thermal pressure (outward) and the weight of the material above pressing downward (inward). This balance is called hydrostatic equilibrium. The equation of hydrostatic equilibrium is given as:

$$\frac{d}{dr}P = -G \cdot \frac{M \cdot \rho}{r^2}$$

The mass conservation equation states how the mass of a star must change with respect to the radius, r. It describes the mass inside a shell of radius r and thickness dr. The total stellar mass is given by summing the mass of each shell.

$$\frac{d}{dr}M = 4 \cdot \pi \cdot r^2 \cdot \rho$$

The energy conservation equation describes how the luminosity of a star must vary with radius. Similar to above, it is helpful to think of the Sun as concentric shells. A star's luminosity is given by summing the energy of each shell. $\varepsilon$ represents the total energy generation rate, which will be discussed later.

$$\frac{d}{dr}L = 4 \cdot \pi \cdot r^2 \cdot \rho \cdot \varepsilon$$

The temperature equation for radiative transport is given by,

$$\frac{d}{dr}T = \frac{-3}{4 \cdot a \cdot c} \cdot \frac{\kappa_{tot} \cdot \rho}{T^3} \cdot \frac{L}{4 \cdot \pi \cdot r^2}$$

where $\kappa_{tot}$ signifies the solar opacity, while

$$\frac{d}{dr}T = -\left(1 - \frac{1}{\gamma}\right) \cdot \frac{\mu \cdot m_H}{k} \cdot \frac{G \cdot M}{r^2}$$

assumes that the temperature gradient is determined by convection (where $\mu$ and $\gamma$ are equal to ~.62 and 5/3 respectively, and k and $m_H$ are the Boltzman constant and the mass of a hydrogen atom (see Appendix C)). This will become important later, but for now it is sufficient to note that the second equation is applied when:

$$\left|\frac{d}{dr}T_{radiative}\right| > \left|\frac{d}{dr}T_{convective}\right|.$$

We also need to find an expression for how density ($\rho$) changes with radius. To do so we need an expression for $\rho$ that we can plug into the equation of hydrostatic equilibrium. For this we will utilize the perfect gas equation, adding to it an expression for radiation pressure.

$$P = \frac{\rho \cdot k \cdot T}{\mu \cdot m_H} + \frac{a \cdot T^4}{3}$$ Perfect Gas Equation + Radiation Pressure

Differentiating with respect to radius, setting the result equal to the equation of hydrostatic equilibrium, and simplifying gives:

$$\frac{d}{dr}\rho = \frac{m_H \cdot \mu}{k \cdot T}\left[\frac{-G \cdot \rho \cdot M}{r^2} - \frac{d}{dr}T\left(\frac{k}{\mu \cdot m_H}\cdot \rho + \frac{4 \cdot a \cdot T^3}{3}\right)\right]$$

We also need an expression for κ, the opacity, and ε, the nuclear generation rate. Let us start by discussing the nuclear generation rate. The mass of any nucleus is less than the sum of the separate masses of its protons and neutrons. Einstein, through his famous equation $E = mc^2$, showed that mass and energy are really two different forms of the same thing. If you were to build nuclei out of individual particles, this missing mass is converted to energy. This process is known as nuclear fusion. The Sun produces the bulk of its energy by nuclear fusion: four hydrogen nuclei are fused to form a single helium nuclei deep within the sun. This chain of reactions is known as the proton-proton chain. The nuclear generation rate of the proton-proton chain, in ergs per gram per second, is given by the following:

$$\varepsilon_{pp}(\rho, T) := 2.38 \cdot 10^6 \cdot \rho \cdot X^2 \cdot f_{pp} \cdot \psi_{pp} \cdot C_{pp} \cdot \left(\frac{T}{10^6}\right)^{\frac{-2}{3}} \cdot \exp\left[-33.80 \cdot \left(\frac{T}{10^6}\right)^{\frac{-1}{3}}\right]$$

The variables $f_{pp} = f_{pp}(X,Y, \rho,T) \approx 1$, $\psi_{pp} = \psi_{pp}(X,Y,T) \approx 1$, and $C_{pp} \approx 1$ are various correction factors which need not be addressed here.[1]

Hydrogen may also be converted to helium through a process known as the CNO cycle. In this cycle, carbon, nitrogen, and oxygen are used as catalysts and are later regenerated. The energy production rate of the CNO cycle is strongly temperature dependent, and thus it favors more massive stars whose central temperature is higher. A good model should produce these results. The CNO cycle's nuclear generation rate, again in ergs per gram per second, is given by:

$$\varepsilon_{CNO}(\rho, T) := 8.67 \cdot 10^{27} \cdot \rho \cdot X \cdot X_{CNO} \cdot C_{CNO} \cdot \left(\frac{T}{10^6}\right)^{\frac{-2}{3}} \cdot \exp\left[-152.28 \cdot \left(\frac{T}{10^6}\right)^{\frac{-1}{3}}\right]$$

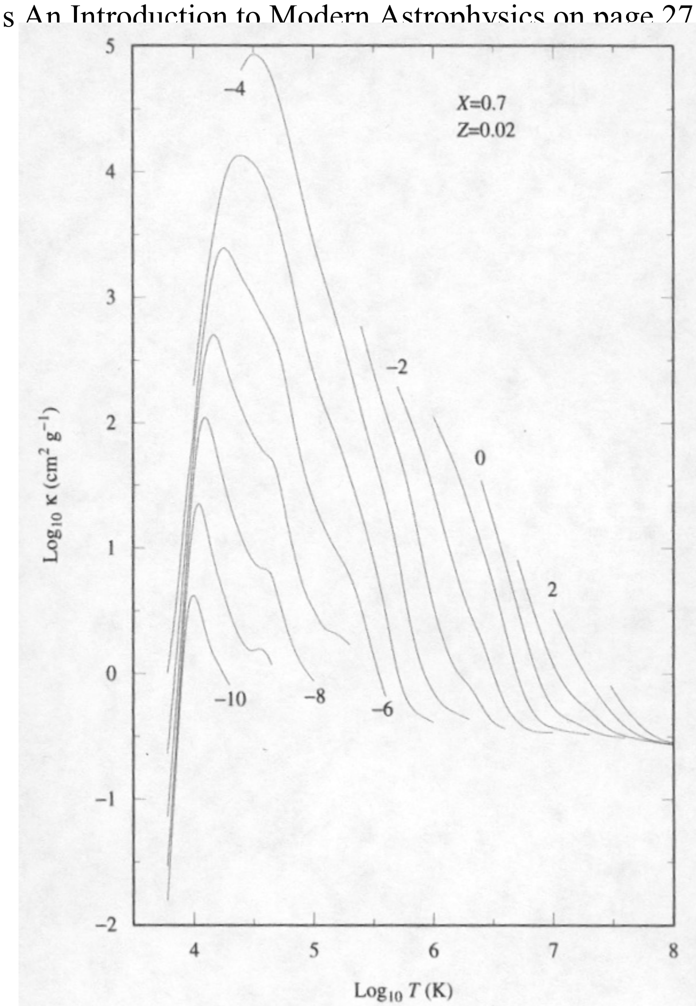
$X_{CNO}$ is the total mass fraction of carbon, nitrogen, and oxygen, and $C_{CNO}$ is another correction term.

To find the total energy generation rate in the Sun, we add $\varepsilon_{pp}$ and $\varepsilon_{CNO}$.

[1] According to Carroll's An Introduction to Modern Astrophysics, "…$f_{pp} = f_{pp}(X,Y, \rho,T) \approx 1$ is the pp [proton-proton] chain screening factor, $\psi_{pp} = \psi_{pp}(X,Y,T) \approx 1$ is a correction factor that accounts for the simultaneous occurrence of PP I, PP II, and PP III, and $C_{pp} \approx 1$ involves higher-order correction terms" (p. 345).

$$\varepsilon(\rho, T) := \varepsilon_{pp}(\rho, T) + \varepsilon_{CNO}(\rho, T)$$

Let's tackle the problem of opacity next. It takes a long time for photons produced by nuclear reactions in the core to reach the surface (on the order of hundreds of thousands of years). In the opaque interior a photon travels only about 1 centimeter before it runs into an atom or ion and is absorbed. A measure of the gas' ability to absorb the photons is called its opacity. You can not see into the interior of a star because the gas has a high opacity. Opacity in the Sun is dependent on both temperature and density. A graph of the "Rosseland" mean opacity (the opacity averaged over frequency) can be found in Carroll's An Introduction to Modern Astrophysics on page 275.




For this model, a program was developed to replicate this graph (see Appendix A for more information).

I noted earlier that convection will take place when, $\left|\frac{d}{dr}T_{radiative}\right| > \left|\frac{d}{dr}T_{convective}\right|$, that is, when the convective temperature gradient is less than the radiative temperature gradient. We can also express this inequality as follows:

$$\frac{\left|\frac{d}{dr}T_{radiative}\right|}{\left|\frac{d}{dr}T_{convective}\right|} > 1$$

This can be incorporated into our model by writing a simple program. First, we formally define the equations governing each temperature gradient.

$$dTdr_{radiative}(r,\rho,L,T) := \frac{-3}{4 \cdot a \cdot c} \cdot \frac{\kappa_{tot}(\rho,T) \cdot \rho}{T^3} \cdot \frac{L}{4 \cdot \pi \cdot r^2}$$

$$dTdr_{convective}(r,M) := -\left(1 - \frac{1}{\gamma}\right) \cdot \frac{\mu \cdot m_H}{k} \cdot \frac{G \cdot M}{r^2}$$

By defining a ratio between $T_{radiative}$ and $T_{convective}$ we can simplify the inequality whose condition must be met for convection to occur.

$$dTdr_{ratio}(r,\rho,M,L,T) := \frac{dTdr_{radiative}(r,\rho,L,T)}{dTdr_{convective}(r,M)}$$

If the ratio is greater than 1, our model should use $T_{convection}$. Therefore, finally:

$$T_{program}(r,\rho,M,L,T) := \begin{cases} dTdr_{radiative}(r,\rho,L,T) & \text{if } dTdr_{ratio}(r,\rho,M,L,T) < 1 \\ dTdr_{convective}(r,M) & \text{if } dTdr_{ratio}(r,\rho,M,L,T) \geq 1 \end{cases}$$

Now that all of the variables have been defined, they can be used to solve a matrix of non-linear, simultaneous, differential equations. These equations will be solved by the Runga-Kutta method found in Mathcad. The matrix D(r,S) is the right hand side of each equation of stellar structure (for ρ, M, L, and T), expressed in components of a vector, S, where r is the independent variable.

$$S = \begin{pmatrix} \rho \\ M \\ L \\ T \end{pmatrix}$$

ρ, M, L, and T are represented below by $S_0$, $S_1$, $S_2$, and $S_3$ respectively.

$$D(r,S) := \begin{bmatrix} \frac{m_H \cdot \mu}{k \cdot S_3}\left[\frac{-G \cdot S_0 \cdot S_1}{r^2} - \left[\frac{-3}{4 \cdot a \cdot c} \cdot \frac{\kappa_{tot}(S_0, S_3) \cdot S_0}{(S_3)^3} \cdot \frac{S_2}{4 \cdot \pi \cdot r^2}\right]\left[\frac{k}{\mu \cdot m_H} \cdot S_0 + \frac{4 \cdot a \cdot (S_3)^3}{3}\right]\right] \\ 4 \cdot \pi \cdot r^2 \cdot S_0 \\ 4 \cdot \pi \cdot r^2 \cdot S_0 \cdot \varepsilon(S_0, S_3) \\ T_{program}(r, S_0, S_1, S_2, S_3) \end{bmatrix}$$

We also need to define initial conditions, that is, the density, mass, luminosity, and temperature at the center of the Sun (r = 0). The mass and luminosity are both equal to 0. The density (g/cm$^3$) and temperature (K) are chosen such that they give the most accurate values of mass and luminosity at $r = R_{sun}$. The initial values are placed in a matrix and passed to the Runga-Kutta function, which will solve the differential equations from 0 to $R_{sun}$, breaking the interval up into N steps.

$$\rho_{center} := 86$$

$$T_{center} := 1.47 \cdot 10^7$$

$$IC := \begin{pmatrix} \rho_{center} \\ 0 \\ 0 \\ T_{center} \end{pmatrix}$$

$$Model := Rkadapt(IC, 0, R_{sun}, N, D)$$

Let's look at some of the results. The density, mass, luminosity, and temperature at $r = R_{sun}$ are:

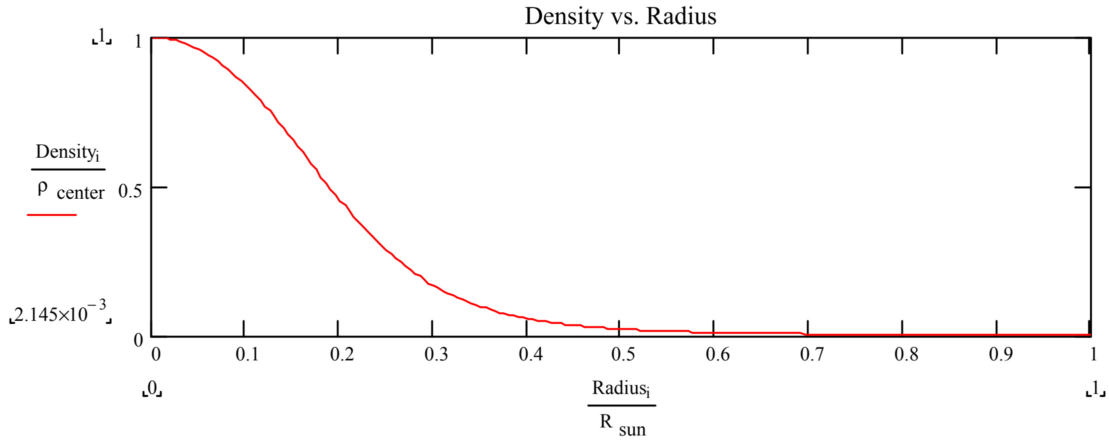
$$Density_N = 0.185 \qquad \frac{Mass_N}{M_{sun}} = 1.271 \qquad \frac{Luminosity_N}{L_{sun}} = 1 \qquad Temperature_N = 6.776 \times 10^6$$

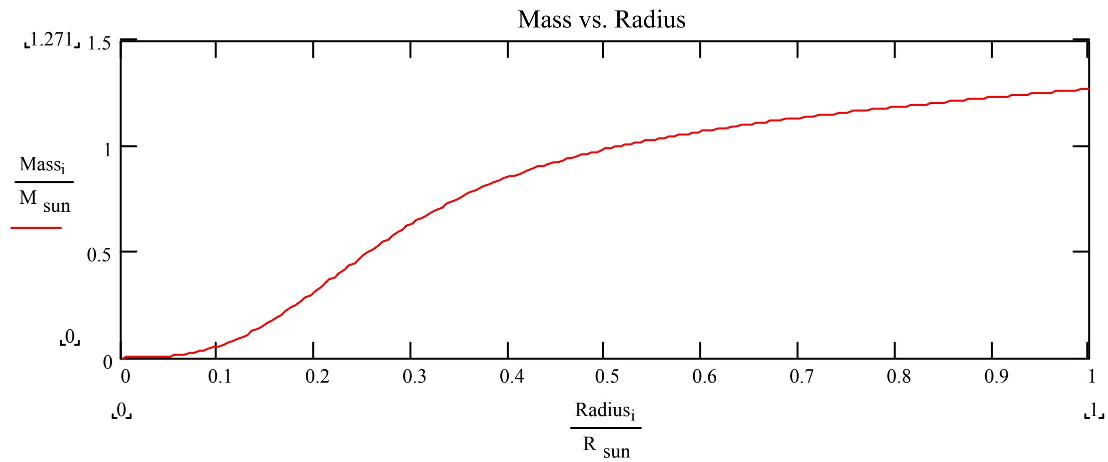
As expected, the density is close to 0 and the luminosity is approximately $L_{sun}$ ($3.826*10^{33}$ ergs/s). The total mass is slightly higher than expected, which is most likely due to the low central density. (More involved simulations place $\rho_{center}$ at approximately 150 g/cm$^3$.) We can find the temperature at the Sun's surface ($T_{effective}$) by assuming the Sun to radiate as a blackbody and using the Stefan-Boltzman equation.

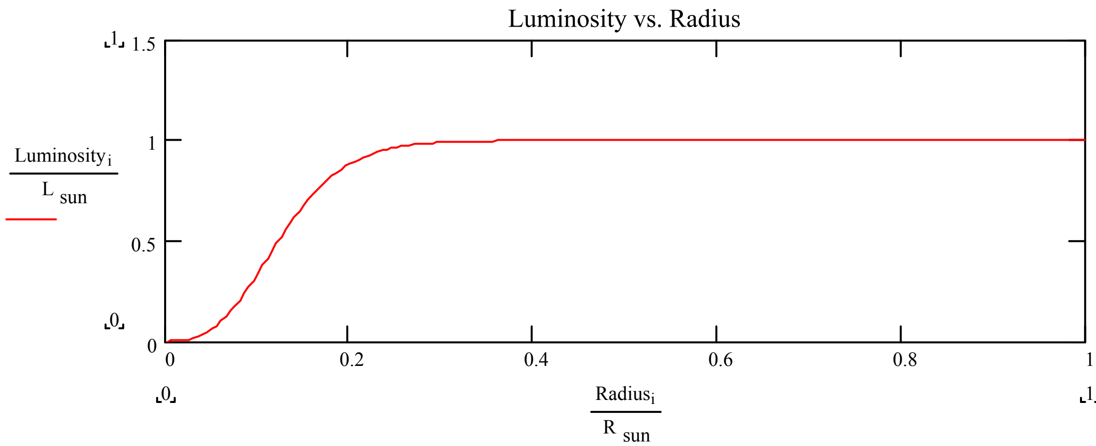
$$T_{effective} := \left[\frac{Luminosity_N}{4\cdot\pi\cdot(R_{sun})^2\cdot\sigma}\right]^{\frac{1}{4}} \qquad T_{effective} = 5.771\times 10^3 \qquad \frac{T_{effective}}{T_{sun}} = 1$$

Our result of 5771K agrees quite well with the accepted value of 5770K. Note that the D-matrix yielded an answer over 1000 times greater than this, an issue which I'll come back to a little later.

It will be instructive to produce graphs of our results to see how each variable changes with radius.

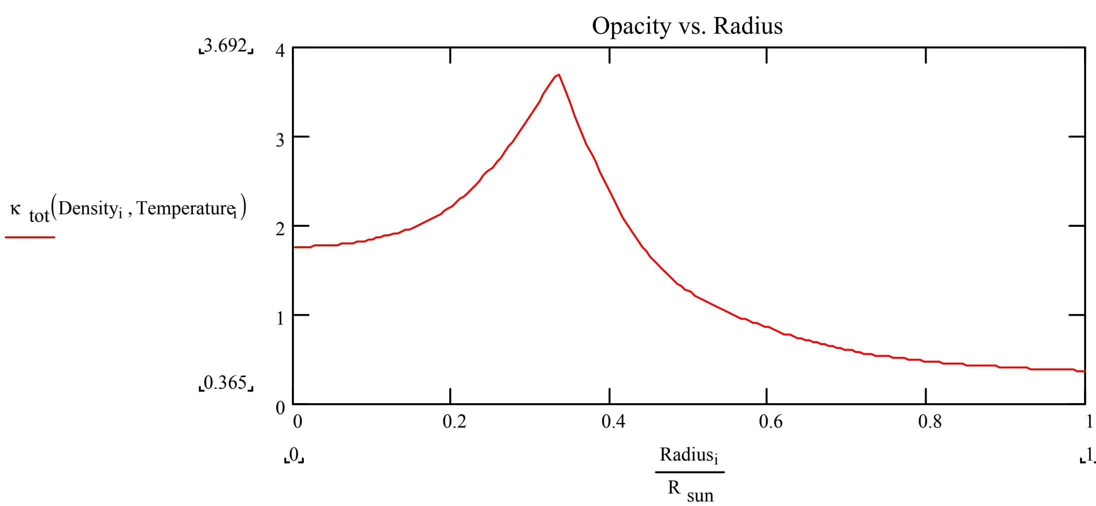




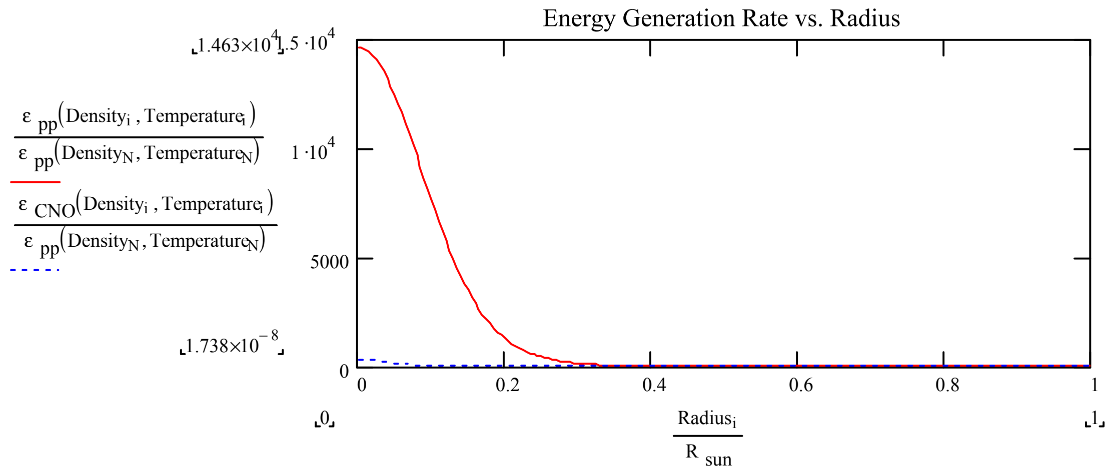

The density decreases rapidly, falling almost to approximately 0 by ½ a solar radius. This makes sense intuitively, since material at the Sun's core will be greatly compressed by that above it. The mass rises quickly near the core, and then more gradually when $.5R_{sun} < r < R_{.sun}$. Note that dM/dr (the change in mass with respect to radius) ≠ 0 at $r = R_{.sun}$, signifying that in our model $r > R_{.sun}$. Luminosity also changes most rapidly near the core, leveling off around ¼ a solar radius.

How does opacity changes with radius?



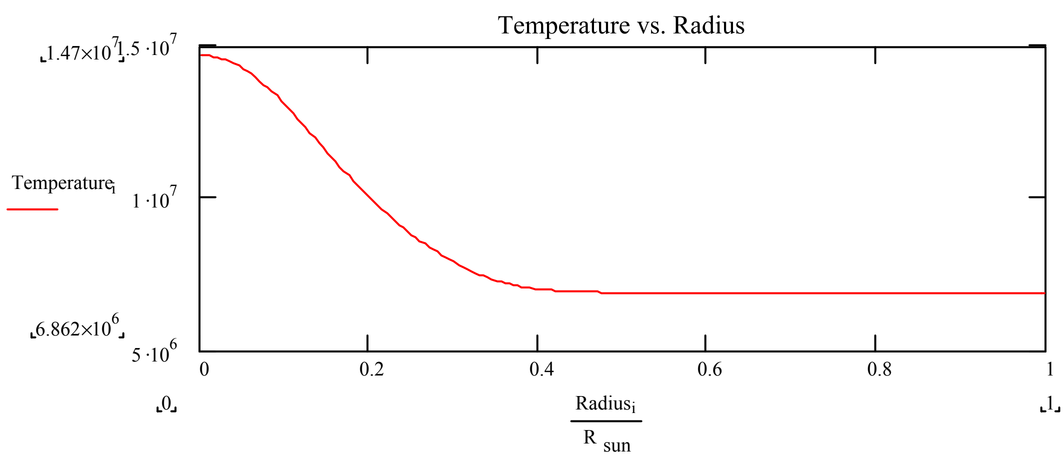

The opacity peaks at $r = .335R_{sun}$, and as expected, decreases as r approaches $R_{sun}$ according to Kramer's law ($\kappa \alpha T^{-3.5}$). The opacity peaks just outside the core due to electron scattering. Free electrons are abundant in this region due to the ionization of hydrogen and helium.

How does the energy generation rate of both the proton-proton chain and the CNO cycle change with radius? The graph shows:




It is easy to see that the proton-proton chain (in red) is dominant over the CNO cycle. However, the CNO cycle does contribute to the total energy generation rate very near the core. It is also interesting to note that nuclear energy generation seems to cease at approximately $r = .4R_{sun}$. This makes sense since both the proton-proton chain and the CNO cycle are temperature dependent, and the temperature drops rapidly outside the core.

Let us address how temperature varies with radius.




The temperature falls as the radius, r, increases, but is over 1000 times the accepted value ($T_{effecive} = 5770K$) at $r = R_{sun}$. To understand this let's take a second to discuss means of energy transport in the sun.

One way to move energy from the interior of a star to its surface is via radiation, that is, photons produced in the core are repeatedly absorbed and re-emitted by stellar atoms, gradually propagating to the surface. A second way is via convection, which is a non-radiative mechanism involving a physical "upwelling" of matter much as in a pot of boiling water. As noted before, convection sets in when:

$$\left|\frac{d}{dr}T_{radiative}\right| > \left|\frac{d}{dr}T_{convective}\right|.$$

According to Carroll's An Introduction to Modern Astrophysics, "…convection will occur when (1) the stellar opacity is large… (2) a region exists where ionization is occurring… (3) the local gravitational acceleration is low… (4) the temperature dependence of the nuclear energy generation rate is large, causing a steep radiative flux gradient and large temperature gradient" (p. 360). Conditions 1, 2, and 3 are met in the atmospheres of stars, while the $4^{th}$ condition is met near the core, where the CNO-cycle is occurring.

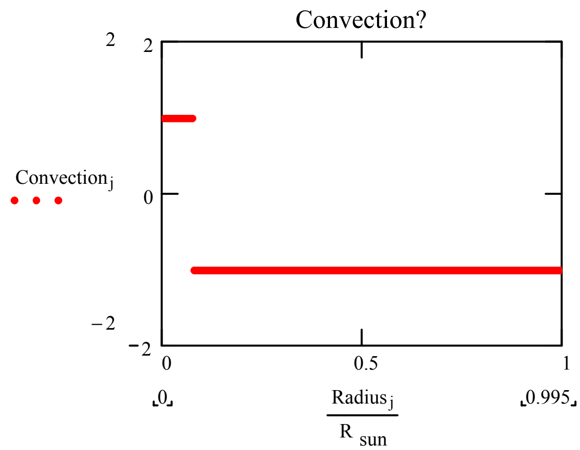
Let us examine if and where convection occurs in our model. The expression:

$$\frac{d(\ln(P))}{d(\ln(T))} < \frac{\gamma}{\gamma - 1} \quad \text{is entirely equivalent to} \quad \left|\frac{d}{dr}T_{radiative}\right| > \left|\frac{d}{dr}T_{convective}\right|.$$

Writing a short program will make this easier to understand. If the condition is met (convection occurs), the program will graph y = 1, while if radiation is the means of transport, it will graph y = –1.

$$Convection_j := if\left(\frac{\ln\left(\frac{P_{j+1}}{P_j}\right)}{\ln\left(\frac{Temperature_{j+1}}{Temperature_j}\right)} < \frac{\gamma}{\gamma - 1}, 1, -1\right)$$

Graphing $Convection_j$ versus radius gives:




The model shows convection occurs until r = $.08R_{sun}$, that is, in the inner 8%. If you recall, convective transport arises near the core (see condition 4) where the CNO cycle occurs. Note that the graph titled "Energy Generation Rate vs. Radius" shows that the CNO cycle is active near the core. Also note the opacity is quite high just outside the core, making it difficult for photons to escape, again leading to convective transport. In our model, however, convection does NOT occur as r approaches $R_{sun}$. This may be the reason why the D-matrix yields such high temperatures as r approaches $R_{sun}$. It is also possible that our crude model of opacity is at fault, and that a more detailed model would lower the temperature.

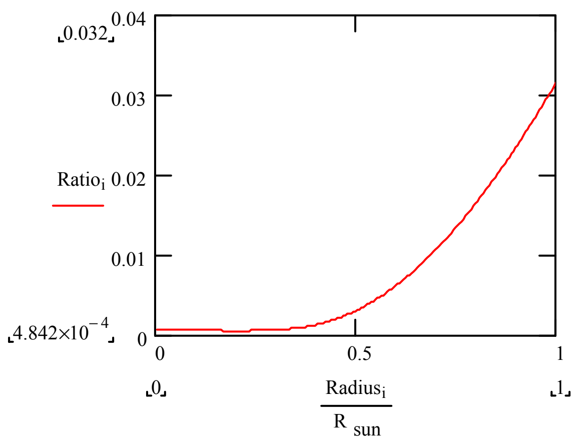
Let's go back and see how much of a role radiation pressure played in the Sun. Recall that:

$$P_i := \frac{k \cdot Density_i \cdot Temperature_i}{\mu \cdot m_H} + \frac{a \cdot (Temperature_i)^4}{3}$$

The first term is just the perfect gas equation, while the second is radiation pressure. A ratio between the two will allow us to measure the significance of each.

$$Ratio_i := \frac{\dfrac{a \cdot (Temperature_i)^4}{3}}{\dfrac{k \cdot Density_i \cdot Temperature_i}{\mu \cdot m_H}}$$

Graphing this ratio versus radius gives:




From the graph it is clear that though radiation pressure increases with radius, it plays little role in the Sun.

Since computers are a very fast way of making errors, in computational work we always need checks. Each calculation was checked using Simpson's Integration Method, and the internal inconsistencies were found to be small. For details, see Appendix B.

## III. THE MOMENT OF INERTIA FOR THE SUN

Now that the model is in place, let's use it to do some interesting calculations. For example, what is the moment of inertia of the Sun?

For a solid rotating sphere, the moment of inertia is found quite simply to be:

$$I = \frac{2}{5} \cdot M \cdot R^2$$

However, this form can not be used in the case of the Sun, since it assumes constant density throughout the body. For the Sun, it becomes necessary to assume a series of concentric spherical shells, each with its own density and therefore own moment of inertia. These shells can be summed over the radius of the Sun to find the moment of inertia for the whole body. The moment of inertia for a spherical shell is:

$$I_{shell} = \frac{2}{3} \cdot M \cdot R^2$$

Each shell will have a different mass (due to the different densities of the shells) and radius.

$$dr := \frac{R_{sun}}{N} \qquad dM_i := 4 \cdot \pi \cdot \left(Radius_i\right)^2 \cdot Density_i \cdot dr \qquad I_i := \frac{2}{3} \cdot dM_i \cdot \left(Radius_i\right)^2$$

The moment of inertia, "I", will then be given by:

$$I := \sum_{i=0}^{N} I_i$$

$$I = 1.485 \times 10^{54}$$

## IV. ANGULAR FREQUENCY?

If the Sun were to collapse into a white dwarf, how fast would it be rotating? To simplify the problem, neglect mass loss and assume the white dwarf is of constant density. Let us tackle this problem using the idea of conservation of angular momentum ($I*\omega$). Simply put, the angular momentum of the Sun must be equal to the angular momentum of the white dwarf it collapses into.

$$I_{wd} \cdot \omega_{wd} = I_{sun} \cdot \omega_{sun}$$

The Sun rotates differentially, that is, the frequency of rotation depends on latitude. Let's assume the frequency to be 26 days, the frequency of rotation along the Sun's equator. Therefore:

$$\omega_{sun} = \frac{2 \cdot \pi}{Period} \qquad \omega_{sun} := \frac{2 \cdot \pi}{2.246 \cdot 10^6} \qquad \omega_{sun} = 2.798 \times 10^{-6}$$

What is the moment of inertia of the white dwarf? Let's assume the dwarf to have the same radius as Sirius B (a well documented case), r = 5.5*10^8 cm. Approximating the dwarf to be a solid rotating sphere, we have:

$$I_{wd} := \frac{2}{5} \cdot M_{sun} \cdot R_{wd}^{\ 2} \qquad I_{wd} = 2.407 \times 10^{50}$$

Therefore:

$$\omega_{wd} = \frac{I_{sun} \cdot \omega_{sun}}{I_{wd}} \qquad \omega_{wd} := \frac{2.013 \times 10^{54} \cdot 2.798 \times 10^{-6}}{2.407 \times 10^{50}} \qquad \omega_{wd} = 0.023$$

To find how quickly the white dwarf is rotating we simply find $1/\omega_{wd}$.

$$\frac{1}{\omega_{wd}} = 42.735$$

This shows that the white dwarf is rotating quite rapidly. It rotates around itself every 43 seconds.

## V. A LARGER STAR?

It would be instructive to apply this model to a large star in which convection and radiation pressure will hopefully play a larger role. The same matrix will apply, only with different initial conditions.

For a 60 solar mass, O5 star (Carroll, Appendix E):

$$M_{big} := 60 \cdot M_{sun} \qquad L_{big} := 790000 \cdot L_{sun} \qquad R_{big} := 15 \cdot R_{sun}$$

Our new initial conditions are:

$$\rho_{center} := 5.8$$

$$T_{center} := 3.8 \cdot 10^7$$

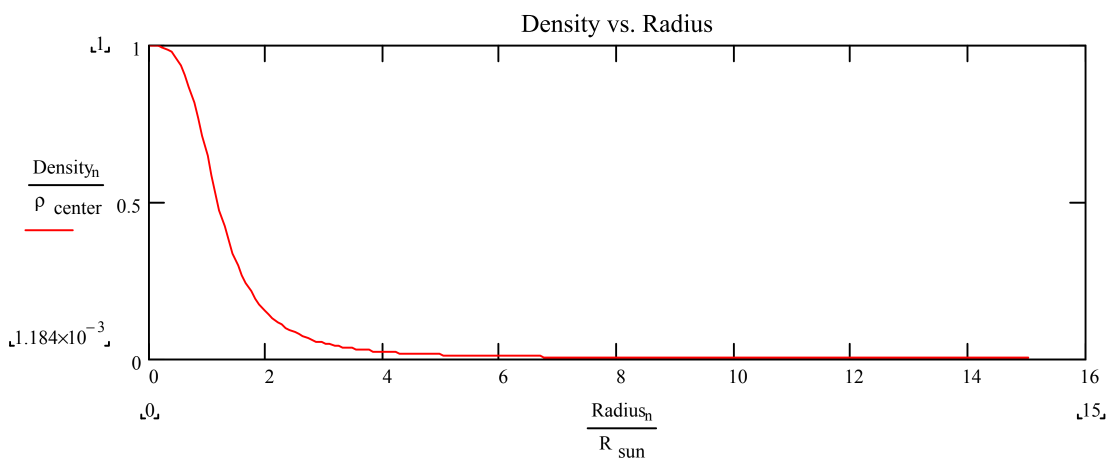
The density, mass, luminosity, and temperature at r = $R_{sun}$ are:

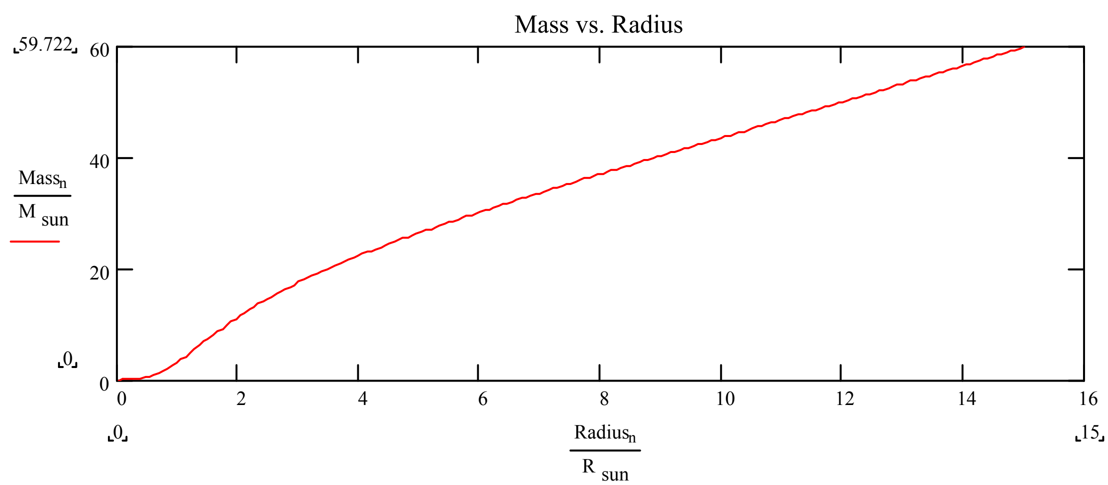
$Density_N = 6.868 \times 10^{-3}$ $\frac{Mass_N}{M_{sun}} = 59.722$ $\frac{Luminosity_N}{L_{sun}} = 9.105 \times 10^4$ $Temperature_N = 3.027 \times 10^7$

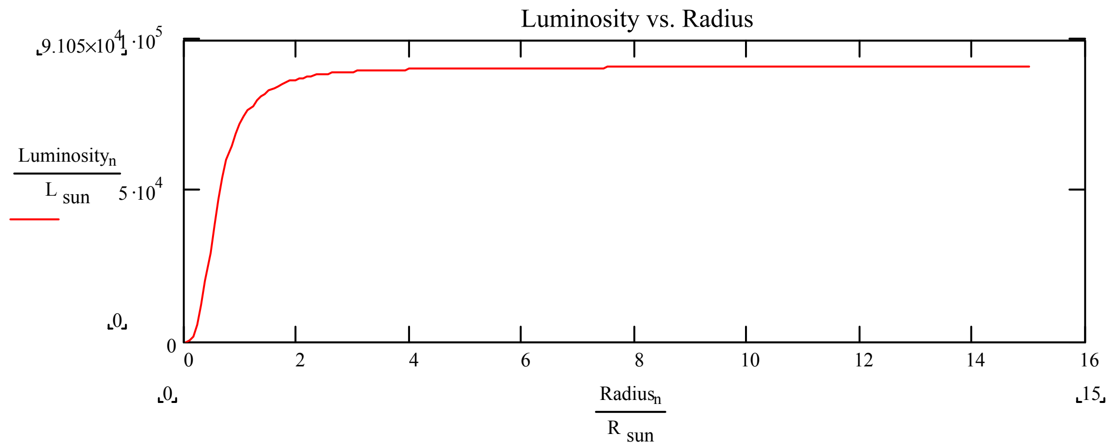
We can again find the temperature at the star's surface ($T_{effective}$) by assuming the star to radiate as a blackbody and using the Stefan-Boltzman equation.

$$T_{effective} := \left[\frac{Luminosity_N}{4 \cdot \pi \cdot (R_{sun})^2 \cdot \sigma}\right]^{\frac{1}{4}} \qquad T_{effective} = 1.002 \times 10^5 \qquad \frac{T_{effective}}{T_{sun}} = 17.371$$

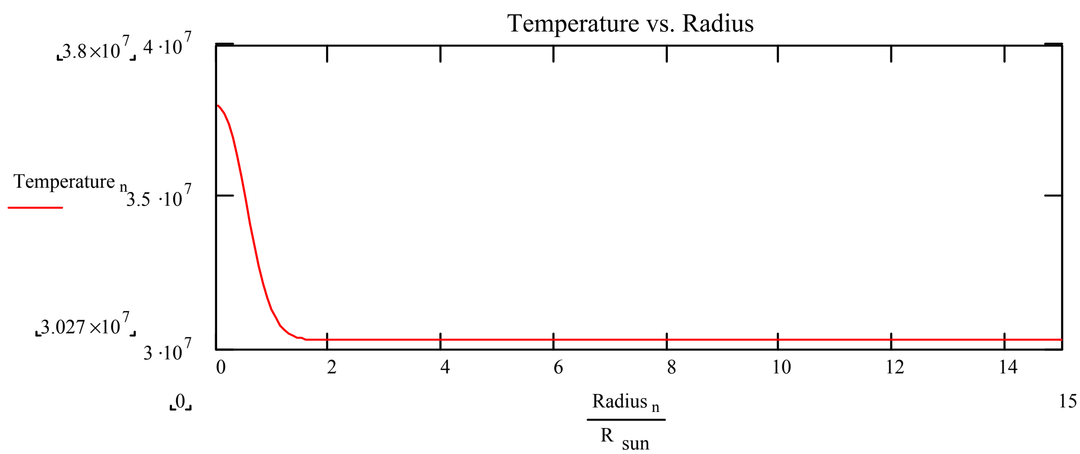
Graphing each we get:



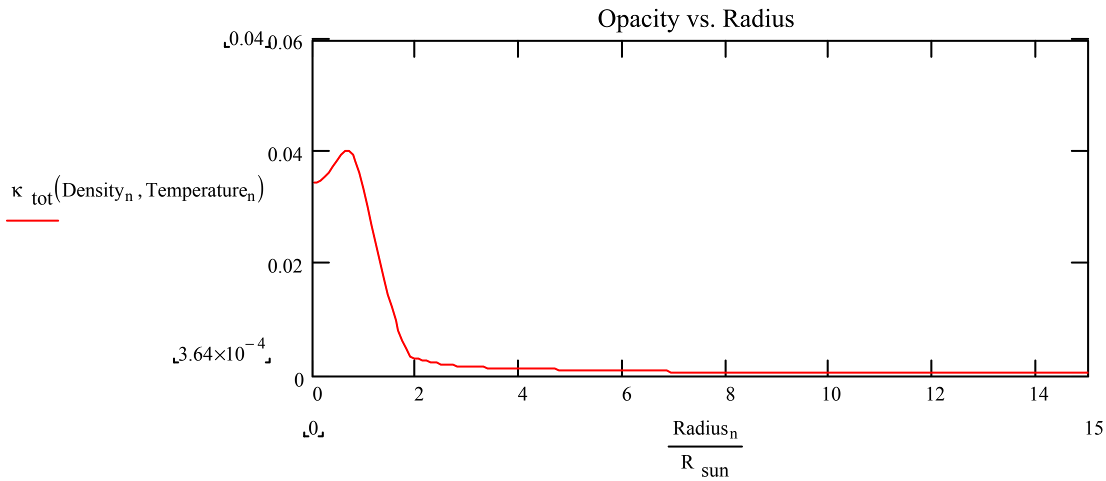



Again, the density approaches 0 as r = $R_{big}$ and the temperature given by the D-matrix is significantly higher than expected. For whatever reason, upon inspection, dM/dr never approaches 0, but actually remains fairly constant at r > $R_{big}$. The luminosity levels off at approximately .15$R_{sun}$, while in the Sun, a significantly smaller star, it increases until approximately .25$R_{sun}$.



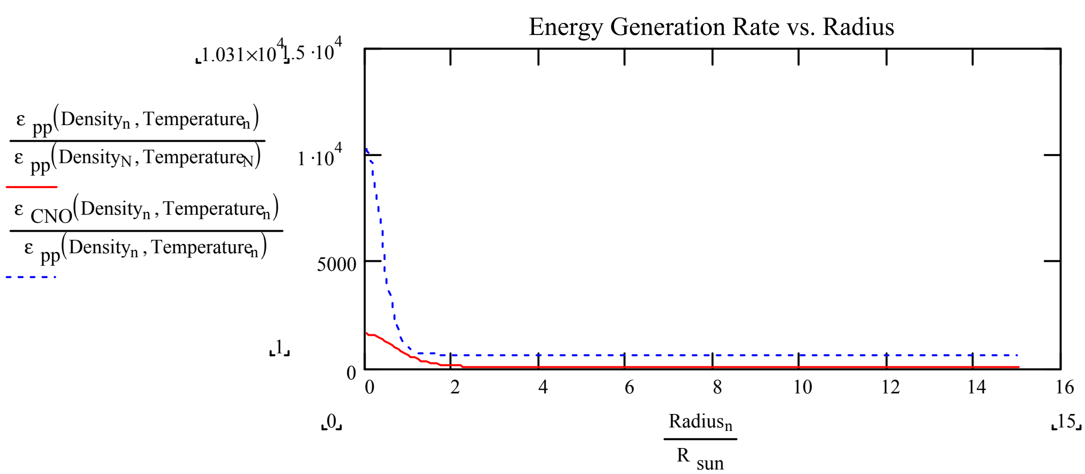

The temperature drops off faster than in the Sun, at approximately r = .07$R_{sun}$ opposed to r = .4$R_{sun}$.

How does the opacity vary with radius in the larger star?



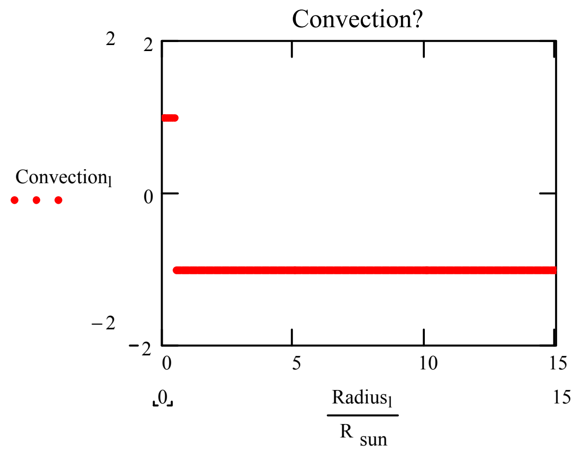

The opacity is significantly lower than the Sun's, signifying that the larger star is more tenuous. It also peaks very near the core of the star.

The energy generation rate is given by:




As expected, the CNO cycle is dominant, most notably in the core where the temperature is the greatest.

Again, let's go back and see how much of a role convection and radiation pressure play in the O5 star. Using the same methods outlined previously, we get:




The model shows convection occurs near the core until r = .04$R_{sun}$, that is, in the inner 4%. This makes sense if we consider the opacity peaks closer to the core. Again, no convection arises elsewhere. This is unfortunate since a larger, tenuous star should be largely convective. Again, the lack of convection may be due to the simple program developed for opacity.

Each calculation was checked using Simpson's Integration Method, and the internal inconsistencies were found to be small. Details can be found in Appendix B.

## VI. REFERENCES


Carroll, Bradley W., and Ostlie, Dale A. An Introduction To Modern Astrophysics. Addison-Wesley Publishing Company, Inc.: Reading, Massachusetts, 1996.

Morse, Andrew. "Superstition and Mystery." ABC News Internet Ventures. Online. Accessed 16 April 2001. Available: http://abcnews.go.com/sections/world/DailyNews/eclipse990810_history.html.


## APPENDIX A:

As noted, a graph of the "Rosseland" mean opacity can be found in Carroll's An Introduction to Modern Astrophysics, on page 275. Each curve (constant density) was fit using a power law, its general form given by:

$$\kappa = \kappa_0 \cdot T^m$$

where

$$m = \frac{\log(\kappa_2) - \log(\kappa_1)}{\log(T_2) - \log(T_1)}.$$

The fits were found to be:

$\kappa_{p3}(T) := 2.37 \cdot 10^{9} \cdot T^{1.25}$

$\kappa_{p2}(T) := 1.78 \cdot 10^{9} \cdot T^{-1.25}$

$\kappa_{p1}(T) := 5.62 \cdot 10^{17} \cdot T^{-2.50}$

$\kappa_{p0}(T) := 3.80 \cdot 10^{25} \cdot T^{-3.75}$

$\kappa_{n1}(T) := 1.33 \cdot 10^{25} \cdot T^{-3.75}$

$\kappa_{n2}(T) := 4.44 \cdot 10^{21} \cdot T^{-3.33}$

$\kappa_{n3}(T) := 1.36 \cdot 10^{22} \cdot T^{-3.57}$

Note that $\kappa_{p3}(T)$ is the opacity at density log(`positive` 3), while $\kappa_{n3}(T)$ is the opacity at density log(`negative` 3).

Linear interpolation was used to insure a smooth curve. In short, it ensures the right mix of opacity expressions when the density falls between two values.

$$\kappa_{p2p3}(\rho,T) := \frac{(\rho - 10^{2}) \cdot \kappa_{p3}(T) + (10^{3} - \rho) \cdot \kappa_{p2}(T)}{10^{3} - 10^{2}} \qquad \text{for } \log(2) < \rho < \log(3)$$

$$\kappa_{p1p2}(\rho,T) := \frac{(\rho - 10^{1}) \cdot \kappa_{p2}(T) + (10^{2} - \rho) \cdot \kappa_{p1}(T)}{10^{2} - 10^{1}} \qquad \text{for } \log(1) < \rho < \log(2)$$

$$\kappa_{p0p1}(\rho,T) := \frac{(\rho - 10^{0}) \cdot \kappa_{p1}(T) + (10^{1} - \rho) \cdot \kappa_{p0}(T)}{10^{1} - 10^{0}} \qquad \text{for } \log(0) < \rho < \log(1)$$

$$\kappa_{n1p0}(\rho,T) := \frac{(\rho - 10^{-1}) \cdot \kappa_{p0}(T) + (10^{0} - \rho) \cdot \kappa_{n1}(T)}{10^{0} - 10^{-1}} \qquad \text{for } \log(-1) < \rho < \log(0)$$

$$\kappa_{n2n1}(\rho,T) := \frac{(\rho - 10^{-2}) \cdot \kappa_{n1}(T) + (10^{-1} - \rho) \cdot \kappa_{n2}(T)}{10^{-1} - 10^{-2}} \qquad \text{for } \log(-2) < \rho < \log(-1)$$

$$\kappa_{n3n2}(\rho,T) := \frac{(\rho - 10^{-3}) \cdot \kappa_{n2}(T) + (10^{-2} - \rho) \cdot \kappa_{n3}(T)}{10^{-2} - 10^{-3}} \qquad \text{for } \log(-3) < \rho < \log(-2)$$

We can now write a program to put it all together:

$$\kappa_{tot}(\rho,T) := \begin{cases} \kappa_{p2p3}(\rho,T) & \text{if } \rho \ge 10^{2} \wedge \rho < 10^{3} \\ \kappa_{p1p2}(\rho,T) & \text{if } \rho \ge 10^{1} \wedge \rho < 10^{2} \\ \kappa_{p0p1}(\rho,T) & \text{if } \rho \ge 10^{0} \wedge \rho < 10^{1} \\ \kappa_{n1p0}(\rho,T) & \text{if } \rho \ge 10^{-1} \wedge \rho < 10^{0} \\ \kappa_{n2n1}(\rho,T) & \text{if } \rho \ge 10^{-2} \wedge \rho < 10^{-1} \\ \kappa_{n3n2}(\rho,T) & \text{if } \rho \ge 10^{-3} \wedge \rho < 10^{-2} \end{cases}$$

**APPENDIX B:**

In computational work checks are very important. In our model, each calculation was checked using Simpson's Integration Method, given by the following:

$$\text{simp}(f, x0, xN, N) := \begin{cases} \text{error("xN must be > x0)")} & \text{if } xN - x0 < 0 \\ \text{error("N must be even.")} & \text{if } (-1)^{N} < 0 \\ \dfrac{xN - x0}{3 \cdot N} \cdot \left[ f_0 + f_N + \displaystyle\sum_{n=1}^{N-1} \text{if}\left[(-1)^{n} < 0, 4, 2\right] \cdot f_n \right] & \end{cases}$$

**For the sun:**

Mass:

$$M_i := 4 \cdot \pi \cdot \text{Density}_i \cdot \left(\text{Radius}_i\right)^2 \qquad M_{simpson} := \text{simp}\left(M, 0, R_{sun}, N\right)$$

$$\frac{M_{simpson} - \text{Mass}_N}{M_{simpson}} = 7.651 \times 10^{-8}$$

Luminosity:

$$L_i := 4 \cdot \pi \cdot \left(\text{Radius}_i\right)^2 \cdot \text{Density}_i \cdot \varepsilon\left(\text{Density}_i, \text{Temperature}_i\right) \qquad L_{simpson} := \text{simp}\left(L, 0, R_{sun}, N\right)$$

$$\frac{L_{simpson} - \text{Luminosity}_N}{L_{simpson}} = -2.733 \times 10^{-7}$$

The temperature is slightly more difficult to check since the model takes into account two means of energy transport, radiation and convection. We'll need to write a short program to toggle between each when necessary. It will be quite similar to the program for temperature defined in the D-matrix.

First, both equations are defined:

$$Ts_{rad_i} := \frac{-3}{4 \cdot a \cdot c} \cdot \frac{\kappa_{tot}\left(\text{Density}_i, \text{Temperature}_i\right) \cdot \text{Density}_i}{\left(\text{Temperature}_i\right)^3} \cdot \frac{\text{Luminosity}_i}{4 \cdot \pi \cdot \left(\text{Radius}_i\right)^2}$$

$$Ts_{conv_i} := -\left(1 - \frac{1}{\gamma}\right) \cdot \frac{\mu \cdot m_H}{k} \cdot \frac{G \cdot \text{Mass}_i}{\left(\text{Radius}_i\right)^2}$$

Once again we define a variable to be the ratio between the two (convection/radiation). Convection is triggered if the ratio > 1.

$$Ts_{ratio_i} := \frac{Ts_{rad_i}}{Ts_{conv_i}}$$

$$Ts_{program_i} := \begin{cases} Ts_{rad_i} & \text{if } Ts_{ratio_i} < 1 \\ Ts_{conv_i} & \text{if } Ts_{ratio_i} \geq 1 \end{cases}$$

Now we can use this program in Simpson's Method just as before.

$$T_{simpson} := \text{simp}\left(Ts_{program}, 0, R_{sun}, N\right) + 1.47 \cdot 10^7$$

$$\frac{T_{simpson} - \text{Temperature}_N}{T_{simpson}} = -2.868 \times 10^{-5}$$

From these calculations it is clear that any internal inconsistencies are small.

**For a 60 solar mass star:**

For a the 60 solar mass star the calculations are identical. Again the inconsistencies are small.

$$\frac{M_{simpson} - Mass_N}{M_{simpson}} = 4.86\times 10^{-9}$$

$$\frac{L_{simpson} - Luminosity_N}{L_{simpson}} = 3.533\times 10^{-5}$$

$$\frac{T_{simpson} - Temperature_N}{T_{simpson}} = 1.47\times 10^{-4}$$

**APPENDIX C:**

**Constants (in cgs):**

Solar mass: $M_{sun} := 1.989 \cdot 10^{33}$

Solar luminosity: $L_{sun} := 3.826 \cdot 10^{33}$

Solar radius: $R_{sun} := 6.9599 \cdot 10^{10}$

Solar effective temperature: $T_{sun} := 5770$

Gravitational constant: $G := 6.67259 \cdot 10^{-8}$

Speed of light: $c := 2.99792458 \cdot 10^{10}$

Boltzmann's constant: $k := 1.380658 \cdot 10^{-16}$

Stefan-Boltzmann constant: $\sigma := 5.6705 \cdot 10^{-5}$

Hydrogen mass: $m_H := 1.673534 \cdot 10^{-24}$

Radiation constant: $a := 7.56591 \cdot 10^{-15}$

Mass fraction of hydrogen in sun: $X := .70$

Mass fraction of helium in sun: $Y := .292$

Mass fraction of metals in sun: $Z := .008$

$$\mu := \left(2 \cdot X + .75 \cdot Y + \frac{Z}{2}\right)^{-1} = .616 \qquad \gamma := \frac{5}{3}$$

Nuclear generation rate:

$\psi_{pp} := 1$ $g_{ff} := 1$ $g_{bf} := 1$ $X_{CNO} := \frac{Z}{2}$

$C_{pp} := 1$ $f_{pp} := 1$ $C_{CNO} := 1$